$$\lambda_1 = \inf_{u \in C_0^\infty} \frac{\int |\nabla u|^2 - Vu^2}{\int u^2}$$

where

$$C_0^\infty(\bar{\Omega}) = \left\{ u : \bar{\Omega} \to \mathbb{R} \;\middle|\; u \in C^\infty(\bar{\Omega}),\ u = 0 \text{ on } \partial\Omega \right\}$$

Thm) Given a Dirichlet Laplace eigenvalue λ_i, we have

$$\lambda_i \geq \lambda_1$$

pf) $\Delta w_i + V w_i + \lambda_i w_i = 0$

$$\Rightarrow 0 = \int -w_i \Delta w_i - V w_i^2 - \lambda_i w_i^2$$

$$= \int |\nabla w_i|^2 - V w_i^2 \, dx - \lambda_i \overset{1}{\int w_i^2}$$

$$\Rightarrow \lambda_i = \frac{\int |\nabla w_i|^2 - V w_i^2}{\int w_i^2} \geq \lambda_1$$

Def) $\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega uv\,dx$

$\Rightarrow \|u\|_{L^2(\Omega)} = \sqrt{\int_\Omega u^2\,dx}$

$= \sqrt{\langle u, u\rangle_{L^2(\Omega)}}$

---

prop) $\lambda_i \neq \lambda_j \Rightarrow \langle w_i, w_j\rangle_{L^2} = 0$.

proof) $\lambda_i w_i = -\Delta w_i - V w_i$

$\lambda_i \langle w_i, w_j\rangle_{L^2} = -\int (\Delta w_i + V w_i) w_j$

$= \int \nabla w_i \nabla w_j - V w_i w_j$

$= \int (-\Delta w_j - V w_j) w_i$

$= \int \lambda_j w_j w_i = \lambda_j \langle w_i, w_j\rangle_{L^2}$

$\lambda_i \neq \lambda_j \Rightarrow \langle w_i, w_j\rangle_{L^2} = 0$.

Fact) There exist Dirichlet Laplace egienpairs $\{(w_i, \lambda_i)\}$

$$\left(\text{i.e. } \begin{array}{ll} \Delta w_i + V w_i + \lambda_i w_i = 0 & \text{in } \Omega \\ w_i = 0 & \text{on } \partial\Omega \end{array} \right)$$

s.t. $\lim \lambda_i = +\infty$. $\langle w_i, w_j \rangle_{L^2} = 0$

$\|w_i\|_{L^2} = 1$. $\lambda_1 \le \lambda_2 \le \cdots$

$\{w_i\}$ spans $L^2(\Omega)$

Remark) The eigenspace of an eigenvalue λ consists of linear combinations of eigenfunctions w/ the eigenvalue λ.

The 2^{nd} eigenvalue

$$\lambda_1 = \inf_{C_0^\infty} \frac{\int |\nabla u|^2 - Vu^2}{\int u^2} = \frac{\int |\nabla w_1|^2 - Vw_1^2}{\int w_1^2}$$

for some $w_1 \in C_0^\infty$ w/ $\|w_1\|_{L^2} = 1$..

↳ "A fact we can't prove in this course.."

Next we define

$$X_1 = \{ u \in C_0^\infty \mid \langle u, w_1 \rangle_{L^2} = 0,\ \|u\|_{L^2} \neq 0 \}$$

and $$\lambda_2 = \inf_{X_1} \frac{\int |\nabla u|^2 - Vu^2}{\int u^2} \geq \lambda_1$$

Fact) $\exists\, w_2 \in X_1$ s.t

$$\lambda_2 = \frac{\int |\nabla w_2|^2 - Vw_2^2}{\int w_2^2}, \quad \|w_2\|_{L^2} = 1$$

Exercise) Show $\Delta w_2 + V w_2 + \lambda_2 w_2 = 0$

i.e. (λ_2, w_2) is an eigenpair.

Idea of proof)

Consider $I(s) = \dfrac{\int |\nabla(w_2 + s\eta)|^2 - V(w_2 + s\eta)^2}{\int (w_2 + s\eta)^2}$

$$\Rightarrow I(s) = \frac{\lambda_2 + 2s\left(\int \nabla w_2 \nabla \eta - V w_2 \eta\right) + C_1 s^2}{1 + 2s \underbrace{\int w_2 \eta}_{\to A} + C_2 s^2}$$

$$= \frac{\lambda_2 + 2s\left(\int \underbrace{-(\Delta w_2 + V w_2)}_{\lambda_2 w_2}\eta\right) + C_1 s^2}{1 + 2As + C_2 s^2}$$

$$= \frac{\lambda_2 + 2\lambda_2 A s + C_1 s^2}{1 + 2As + C_2 s^2}$$

$$= \lambda_2 + \frac{(C_1 - \lambda_2 C_2) s^2}{1 + 2As + C_2 s^2}$$

$\Rightarrow I'(0) = 0.$

we recall that $\langle w_1, w_2 \rangle_{L^2} = 0$.
($\because w_2 \in X_1$)

Similarly,

$$X_2 \triangleq \{u \in X_1 \mid \langle u, w_2 \rangle_{L^2} = 0, \|u\|_{L^2} \neq 0\}$$

$$\lambda_3 = \inf_{X_2} \frac{\int |Du|^2 - Vu^2}{\int u^2}.$$

Fact) $\exists\, w_3 \in X_2$, s.t

$$\lambda_3 = \frac{\int |Dw_3|^2 - Vw_3^2}{\int w_3^2}, \quad \|w_3\|_{L^2} = 1.$$

Then, in the same manner
(w_3, λ_3) is an eigenpair desired

Fact) Iterating this process yield the seq. of pairs

Hilbert space.

Def) Hilbert space is a Banach space equipped w/ an inner product.

Example) L^2 is a Hilbert space.

$$\langle f, g \rangle_{L^2} = \int f g \, dx$$

$$\|f\|_{L^2} = \sqrt{\langle f, f \rangle_{L^2}}$$

Def) [Weak derivative]

Suppose $u \in L^1_{loc}(\Omega)$ and $v \in L^1_{loc}(\Omega)$ satisfy

$$\int_\Omega u \, \partial_i \varphi = - \int_\Omega v \varphi$$

for all $\varphi \in C_0^\infty(\Omega)$. Then, we say v is a weak derivative of u, $v = \partial_i u$.

Def) $H^1(\Omega)$ consists of 1^{st} order weakly differentiable functions u such that

$$\int |\nabla u|^2 + \int u^2 < +\infty$$

i.e. all weak derivative $\partial_i u$, and u belong to $L^2(\Omega)$.

In particular. $u, v \in H^1$

$$\langle u, v\rangle_{H^1} = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv$$

$$\|u\|_{H^1} = \left(\int_\Omega |\nabla u|^2 + \int_\Omega u^2\right)^{1/2}.$$

proposition) H^1 is a Hilbert space.

pf) We only need to show that
H^1 is complete.

Indeed, L^2 is complete.

Let $\{u_m\} \subset H^1$ be a Cauchy seq.

Then. $\lim_{m\to\infty} \|\partial_i u_m - v_i\|_{L^2} = 0$ — (*)

$\lim_{m\to+\infty} \|u_m - u\|_{L^2} = 0$.

$v_1, \cdots, v_n, u \in L^2$

Claim: $v_i = \partial_i u$ is a weak derivative.

$\Leftrightarrow \int v_i \varphi = -\int u \varphi_i \quad \forall \varphi \in C_0^\infty$

True by (*).

Def) H_0^1 is a closure of $C_0^\infty \cap H^1$.

Namely, $u \in H_0^1$

$\Leftrightarrow \exists \{u_m\} \in C_0^\infty$ s.t

$$\lim_{\min\{m,k\} \to +\infty} \|u_m - u_k\|_{H^1} = 0$$

prop) $H_0^1 \subset H^1$

pf) H^1 is complete

$\Rightarrow$ The limit of any cauchy seq belongs to H^1.

prop) H_0^1 is a Hilbert space

Thm) $\exists\, w_1 \in H_0^1$ s.t. $\|w_1\|_{L^2} = 1$

$$\lambda_1 = \frac{\int |Dw_1|^2 - V w_1^2}{\int w_1^2}$$

Def) $\{u_k\} \subset H^1$ converges $u \in H^1$

WEAKLY, if

$$\lim_{k \to +\infty} \langle u_k, v \rangle_{H^1} = \langle u, v \rangle_{H^1}$$

for every $v \in H_0^1$

and we denote $u_k \rightharpoonup u$ in H^1

Thm) [weak convergence of bounded sequence]

If $\{u_k\} \subset H^1$ and $\|u_k\|_{H^1} \leq C$ for some C independent of k,

then $\exists$ a subsequence $\{u_{k_j}\}$ and $u \in H^1$ s.t.

$$u_{k_j} \longrightarrow u \quad \text{in } H^1_0$$

In particular, if $\{u_k\} \subset H^1_0$ then $u \in H^1_0$.

Thm) [Rellich - Kondrach Compactness]

If $\{u_k\} \in H^1_0$ satisfy $\|u_k\|_{H^1} \leq C$.

then $\exists$ a subsequence u_{k_j} s.t.

$\lim_{k_j \to +\infty} \|u_{k_j} - u\|_{L^2} = 0$ for some $u \in L^2$.

$\Leftrightarrow u_{k_j} \longrightarrow u$ in L^2

proof of existence of minimizer (w_t) in H_0^1..

$\exists \{u_k\} \in C_0^\infty$ s.t. $\|u_k\|_{L^2} = 1$

$$J(u_k) = \int |\nabla u_k|^2 - V u_k^2 \, dx$$

$$\lim_{k \to +\infty} J(u_k) = \lambda_1$$

$$\Rightarrow \|u_k\|_{H^1}^2 = \int |\nabla u_k|^2 + u_k^2$$

$$= J(u_k) + 1 + \int V u_k^2$$

$$\leq J(u_k) + 1 + \sup V.$$

$\Rightarrow \|u_k\|_{H^1}$ is uniformly bounded.

By the two convergence theory..

$\exists$ a subsequence u_{k_j} and $u \in H_0^1$

$u_{k_j} \rightharpoonup u$ in H^1. $u_{k_j} \to u$ in L^2..

$$\int V u^2 - \int V u_{k_j}^2 = \int V(u-u_{k_j})(u+u_{k_j})$$

$$\leq \sqrt{\int V^2 (u+u_{k_j})^2 \int (u-u_{k_j})^2}$$

$$\leq (\sup |V|)\, \|u+u_{k_j}\|_{L^2} \|u-u_{k_j}\|_{L^2}$$

$$\leq C\,(\underbrace{\|u\|_{L^2}}_{=1} + \underbrace{\|u_{k_j}\|_{L^2}}_{=1})\,\underbrace{\|u-u_{k_j}\|_{L^2}}_{\to 0}$$

$$\Rightarrow \lim_{k_j \to +\infty} \int V u_{k_j}^2 = \int V u^2$$

Claim: $\int |\nabla u|^2 \leq \liminf_{k_j \to +\infty} \int |\nabla u_{k_j}|^2$

If the claim is true.

$$\int |\nabla u|^2 - V u^2 \leq \liminf \int |\nabla u_{k_j}|^2 - V u_{k_j}^2$$

$$= \liminf J(u_k) = \lambda_1$$

$$\Rightarrow \lambda_1 = \frac{\int |\nabla u|^2 - V u^2}{\int u^2} \Rightarrow w_1 = u \in H_0^1.$$

$$\|u\|_{H^1}^2 = \langle u, u\rangle_{H^1} = \lim \langle u, u_{k_j}\rangle_{H^1}$$

$$\langle u, u_{k_j}\rangle_{H} \leq \|u\|_{H^1} \|u_{k_j}\|_{H^1}$$

$$\Rightarrow \liminf \langle u, u_{k_j}\rangle \leq \|u\|_{H^1} \liminf \|u_{k_j}\|_{H^1}$$

$$\Rightarrow \|u\|_{H^1}^2 \leq \|u\|_{H^1} \liminf \|u_{k_j}\|_{H^1}$$

Since $\|u\|_{H^1} \geq \|u\|_{L^2} = 1 > 0.$

$$\Rightarrow \|u\|_{H^1} \leq \liminf_{k_j \to +\infty} \|u_{k_j}\|_{H^1}.$$

$\blacksquare$